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## UNSTEADY STAGNATION-POINT HEAT TRANSFER

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## SUMMARY

An analysis is made of the unsteady, forced-convection heat transfer at a stagnation point whose surface temperature varies arbitrarily with time. The flow is steady and laminar. The first step in the analysis yields the heat-transfer response to a sudden change (step function) in wall temperature, and this is then generalized by a superposition technique to apply to arbitrary variations. Use of the generalized results is illustrated by application to the case where the surface temperature varies linearly with time. Comparison is made between the unsteady-heat-transfer results of this analysis and those computed under the assumption of quasi-steady conditions. Numerical results are presented for a Prandtl number of 0.7 (i.e., gases).

## INTRODUCTION

In a number of important technical applications, it is necessary to compute the forced-convection heat transfer from a surface whose temperature is changing with time. To solve such problems by a direct attack on the governing differential equations (conservation laws) is normally an exceedingly formidable task. As a consequence, it has been customary to simplify matters by supposing that, at each and every moment, there exists an instantaneous steady state. Under such an assumption, the steady-state relations for the heat-transfer coefficient are used in conjunction with the instantaneous temperature difference to compute a heat-transfer rate. The phrase quasi-steady is usually applied to describe the situation in which the transient passes through a sequence of instantaneous steady states.

In reality, there is always a difference between the actual instantaneous heat transfer and the quasi-steady value, the extent of which depends upon the rapidity of the temperature changes and on the response characteristics of the flow. During the initial stages of a transient process or under conditions of very rapid temperature change, it is not expected that the state will be quasi-steady. While nonquasi-steady

situations can exist in both laminar and turbulent flows, they are more likely in the laminar case because of the relatively slower response of such a flow.

The alternative to invoking the quasi-steady assumption is to start with the governing differential equations and solve for the entire time-history of the heat-transfer transient from its beginning to its end. A promising beginning along these lines has been made for internal flow in tubes and flat ducts; solutions of the unsteady energy equation have been obtained for the condition of fully developed, steady velocity distributions, both laminar and turbulent (refs. 1 to 4). For external flow, which is the area of interest here, the problem is more difficult and analysis has been confined to computing small deviations from quasi-steady heat transfer when the state is not quite quasi-steady (refs. 5 to 7). Consideration has been given only to laminar flow.

In the present investigation, attention is directed to the unsteady heat transfer in a laminar stagnation-point flow. The goal is to determine the complete time-history of the heat-transfer transient associated with an arbitrary time-variation of the surface temperature. The analysis is carried out for steady flow of an incompressible, constant-property fluid with negligible viscous dissipation. Also, the wall temperature is spatially uniform at any instant of time. The first step in the study is to determine the heat-transfer response to a sudden change (step function) in wall temperature, starting from an initial condition of no heat transfer ( $T_w = T_\infty$ ). The step-function result serves as a fundamental solution, since by a superposition technique it is generalized to apply for arbitrary time-variations in wall temperature. The initial conditions are also generalized to permit the transient to begin either from a condition of steady-state heat transfer ( $T_w \neq T_\infty$ ) as well as a no-heat-transfer situation ( $T_w = T_\infty$ ). To demonstrate the use of the generalized results, a heat-transfer computation is carried out for the case where the wall temperature varies linearly with time, that is, a ramp function. Comparison is made between the unsteady-heat-transfer results of the present theory and those computed under the quasi-steady assumption. In the final section of the report, a procedure is discussed for estimating unsteady heat transfer under conditions of variable properties and viscous dissipation.

Although the theoretical development may apply to any Prandtl number  $Pr$ , the numerical computations have been carried out for  $Pr = 0.7$  (i.e., gases).

#### SYMBOLS

- $A^*$             proportionality constant, see eq. (28)
- $c_p$             specific heat at constant pressure

h	heat-transfer coefficient, $q/(T_w - T_\infty)$
k	thermal conductivity
Pr	Prandtl number, $c_p \mu / k$
q	heat-transfer rate per unit area at surface
T	static temperature
t	time
t*	dummy integration variable
$U_\infty$	free-stream velocity
u	velocity component in x-direction
$u_1$	proportionality constant, see eq. (3b)
v	velocity component in y-direction
x	coordinate measuring distance along surface
y	coordinate measuring distance normal to surface
$\alpha$	thermal diffusivity, $k/\rho c_p$
$\Delta$	thermal boundary-layer thickness
$\delta$	velocity boundary-layer thickness
$\eta$	dimensionless y-coordinate, $y/\delta$
$\lambda$	Pohlhausen parameter, $\delta^2 u_1 / \nu$
$\mu$	absolute viscosity
$\nu$	kinematic viscosity
$\xi$	dummy integration variable
$\rho$	density
$\tau$	dimensionless time, $u_1 t$
$\tau^*$	dummy integration variable
$\phi$	dimensionless boundary-layer thickness, $\Delta/\delta$

## Subscripts:

inst	instantaneous
qs	quasi-steady
ss	steady state
w	surface
$\infty$	free stream
0	at time 0
$\tau$	at time $\tau$
$\tau-0.855$	at time $\tau - 0.855$
$\tau^*$	function of $\tau^*$
$\tau-\tau^*$	function of $\tau - \tau^*$

## STEP CHANGE IN WALL TEMPERATURE

## Analysis

The starting point for this study of the transient heat-transfer response to a step change in wall temperature is the conservation-of-energy principle. The mathematical statement of this law appropriate to unsteady heat transfer in a constant-property, incompressible, laminar-boundary-layer flow with negligible viscous dissipation is

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (1)$$

Integrating with respect to  $y$  gives the over-all energy balance

$$\frac{\partial}{\partial t} \int_0^{\Delta} (T - T_{\infty}) dy + \frac{\partial}{\partial x} \int_0^{\Delta} u(T - T_{\infty}) dy = -\alpha \left( \frac{\partial T}{\partial y} \right)_{y=0} \quad (2)$$

where  $\Delta$  is the thickness of the thermal boundary layer. This integrated energy equation will now be used in determining the unsteady temperature distribution and heat transfer in a stagnation-point flow. Consideration will be given here to the situation where the transient begins from an equilibrium condition where there is no heat transfer ( $T_w = T_{\infty}$ ).

In a later section, generalization will be made to include transients which begin from an already established steady-state situation ( $T_w \neq T_\infty$ ).

In order to attack equation (2), it is first necessary to know the distribution of the velocity  $u$  across the boundary layer. A consequence of the constancy of the fluid properties is that the velocity distribution is not influenced by the temperature problem, and, as a result, the usual steady-state solution for stagnation-point boundary-layer flow can be used here. An excellent polynomial representation for  $u$ , derived by the Kármán-Pohlhausen method (ref. 8), takes the following form:

$$\frac{u}{U_\infty} = 2\eta - 2\eta^3 + \eta^4 + \frac{\lambda}{6} \left[ \eta(1 - \eta)^3 \right] \quad (3a)$$

where

$$\eta = y/\delta, \quad U_\infty = u_1 x, \quad \lambda = \delta^2 u_1 / \nu \quad (3b)$$

The velocity boundary-layer thickness is denoted by  $\delta$ , while  $\lambda$  is the Pohlhausen parameter and  $u_1$  is a proportionality constant. For stagnation-point flow, reference 8 gives

$$\lambda = 7.052 \quad \text{so that} \quad \delta = (7.052 \nu / u_1)^{1/2} \quad (3c)$$

Not only is equation (3a) a good representation within the velocity boundary layer (i.e., for  $y \leq \delta$ ); but it also serves well for a range outside the velocity boundary layer, where  $u/U_\infty$  is supposed to be unity. For example, at  $y/\delta = 1.1$ , equation (3a) gives  $u/U_\infty = 1.002$ ; while, at  $y/\delta = 1.18$ , which represents the largest value of interest here, equation (3a) gives 1.01. Finally, it may be worthwhile to reiterate the well-established fact that the velocity boundary-layer thickness for a stagnation-point flow is independent of  $x$ , as is seen from equation (3c).

Thus, with an accurate velocity distribution available, consideration may now be given to solving equation (2). This integral equation may be attacked by writing the temperature distribution as a polynomial which satisfies the essential boundary conditions:

$$\frac{T - T_\infty}{T_w - T_\infty} = 1 - \frac{3}{2} \left( \frac{y}{\Delta} \right) + \frac{1}{2} \left( \frac{y}{\Delta} \right)^3 \quad \begin{matrix} t > 0 \\ 0 \leq y \leq \Delta \end{matrix} \quad (4)$$

The thermal boundary-layer thickness  $\Delta$  is an unknown function of time but is independent of  $x$  since  $T_w$  is spatially uniform.<sup>1</sup> The

<sup>1</sup>This will also be true when  $T_w - T_\infty \sim x^n$  in a stagnation-point flow.

dependence of  $\Delta$  upon time will be determined by satisfying the conservation of energy equation (2). When equations (3) and (4) are introduced into equation (2) and the integrations are carried out, there results the following first-order ordinary differential equation for  $\Delta$ :

$$\frac{3}{8u_1} \frac{d\Delta}{dt} + \frac{\Delta^2}{\delta} \left( \frac{1}{5} + \frac{\lambda}{60} \right) - \frac{\Delta^3}{\delta^2} \left( \frac{\lambda}{48} \right) + \frac{\Delta^4}{\delta^3} \left( -\frac{3}{70} + \frac{3\lambda}{280} \right) + \frac{\Delta^5}{\delta^4} \left( \frac{1}{80} - \frac{\lambda}{480} \right) = \left[ \frac{3\nu}{2(\text{Pr})u_1} \right] \frac{1}{\Delta} \quad (5)$$

To complete the statement of the problem, it is necessary to give the initial condition on  $\Delta$ . For the situation where the transient begins from a condition of no heat transfer, all the fluid initially possesses the temperature  $T_\infty$  and, as a result,

$$\Delta = 0 \quad \text{at} \quad t = 0 \quad (6)$$

By using equation (3c),  $u_1$  may be eliminated in favor of  $\delta$ , and  $\lambda$  may be evaluated as 7.052. For computational convenience, new variables  $\phi$  and  $\tau$  are introduced by the definitions

$$\phi = \Delta/\delta \quad \tau = u_1 t \quad (7)$$

and it is noted that, since  $\delta$  is constant, the time variations of  $\phi$  and  $\Delta$  are identical. With these modifications and using the value of 0.7 for the Prandtl number (gases), equation (5) becomes

$$\frac{\phi \, d\phi}{\phi^6 - 14.9202 \phi^5 + 67.0343 \phi^4 - 144.882 \phi^3 + 138.646} = 0.00584444 \, d\tau \quad (8)$$

with the boundary condition

$$\phi(0) = 0 \quad (9)$$

Once equation (8) has been solved for  $\phi$ ,  $\Delta$  is known, and the temperature distribution (4) may be used for carrying out the heat-transfer computation.

It is possible to obtain a closed-form analytical solution of equation (8) by using a partial fraction expansion. The first step is to find the six roots of the polynomial which appears in the denominator. With these, and with considerable algebraic manipulation as outlined in the appendix, equation (8) can be written as

$$\left( \frac{0.000212295}{\varphi - 9.44638} - \frac{0.00399594}{\varphi - 1.17705} + \frac{0.00305397 \varphi - 0.0107341}{\varphi^2 - 5.37067 \varphi + 15.2053} + \frac{0.000729679 \varphi - 0.00218670}{\varphi^2 + 1.07320 \varphi + 0.820075} \right) d\varphi = 0.00584444 d\tau \quad (10)$$

Inspection of the left-hand side of this equation shows that all terms lead to elementary integrals. Carrying out the integration and imposing the boundary condition (9) provide the following relation between  $\varphi$  and  $\tau$  (i.e., between  $\Delta$  and  $t$ ):

$$\begin{aligned} \tau = & -0.0363243 \ln \left( \frac{9.44638}{9.44638 - \varphi} \right) + 0.683717 \ln \left( \frac{1.17705}{1.17705 - \varphi} \right) + \\ & 0.261271 \ln \left( \frac{\varphi^2 - 5.37067 \varphi + 15.2053}{15.2053} \right) + \\ & 0.0624250 \ln \left( \frac{\varphi^2 + 1.07320 \varphi + 0.820075}{0.820075} \right) + \\ & 0.153295 \left[ \tan^{-1} \left( \frac{5.37067 - 2\varphi}{5.65482} \right) - \tan^{-1} 0.949751 \right] - \\ & 0.604741 \left[ \tan^{-1} \left( \frac{2\varphi + 1.07320}{1.45896} \right) - \tan^{-1} 0.735591 \right] \quad (11) \end{aligned}$$

Equation (11) provides the dimensionless time  $\tau$  at which a given dimensionless thickness  $\varphi$  is achieved during the course of development of the thermal boundary layer. A plot, based on equation (11), showing the growth of the thermal boundary layer is presented as the solid curve in figure 1. Certain interesting properties of the solution are discussed below.

First, at very early times, when  $\varphi$  is exceedingly small (i.e.,  $\Delta \ll \delta$ ), it would be expected that heat conduction would be relatively very important compared to energy convection. This intuitive feeling is supported by equation (8), which shows that the conductive term - the constant in the denominator - dominates over all the other terms when  $\varphi$  is small. So, for small times, equation (8) becomes

$$\frac{\varphi d\varphi}{138.646} = 0.00584444 d\tau$$

from which it follows that

$$\varphi = 1.27303 \tau^{1/2} \quad (12)$$

This asymptote is plotted as a straight dotted line on figure 1. For all  $\tau < 0.145$ , the dotted line agrees with the solid curve (exact solution of eq. (8)) to within 2 percent or better, the deviation becoming smaller as  $\tau$  decreases.

At the other extreme, it is expected that, after a long time has passed following the application of the step jump in wall temperature, a steady-state heat-transfer situation will be established. The steady-state solution is obtained by setting  $d\Delta/dt = 0$  in equation (5) or, alternately, setting  $d\phi/dt = 0$  in equation (8). From this, it is found that

$$\phi_{ss} = 1.17705 \quad (13)$$

Inspection of the transient solution (eq. (11)) indicates that the steady state is approached asymptotically as time grows larger and larger. However, all practical effects of the transient (e.g., all significant heat-transfer variations) are over in a finite time.

Thus, with this solution for  $\phi$  (and hence  $\Delta$ ), the temperature distribution (eq. (4)) is known and attention can now be turned to the computation of the heat transfer.

#### Heat-Transfer Results for a Step Change

The instantaneous heat flux at the surface may be calculated by applying Fourier's law:

$$q = -k \left( \frac{\partial T}{\partial y} \right)_{y=0} \quad (14)$$

By utilizing the temperature distribution (eq. (4)) and introducing dimensionless variables, the expression for  $q$  becomes

$$q = \frac{3}{2} \frac{k}{\Delta} (T_w - T_\infty) = \left[ \frac{3}{2} \frac{k}{\delta} (T_w - T_\infty) \right] \frac{1}{\phi} \quad (15)$$

or

$$\frac{q(\nu/u_1)^{1/2}}{k(T_w - T_\infty)} = \frac{0.5649}{\phi} \quad (15a)$$

The heat transfer is seen to depend inversely on  $\phi$ ; and, as a consequence,  $q$  achieves very high values immediately following the step jump and then decreases monotonically with time.

A convenient representation of the heat transfer may be achieved by introducing the steady-state results. By noting that  $\phi_{ss} = 1.17705$ , equation (15a) yields

$$\frac{q_{ss}(\nu/u_1)^{1/2}}{k(T_w - T_\infty)} = 0.4799 \quad (16)$$

Then, the ratio of this equation with (15a) leads to the simple form

$$\frac{q}{q_{ss}} = \frac{1.177}{\phi} \quad (17)$$

This quotient of the instantaneous to the steady-state heat transfer has been plotted as a function of time in figure 2 by making use of the previously determined values of  $\phi$ . As expected, there are very high heat-transfer rates at early times since the thermal boundary layer is thin. An analytical representation appropriate to early times may be obtained by utilizing the asymptotic expression (eq. (12)) for  $\phi$ . With this, equation (17) becomes

$$\frac{q}{q_{ss}} = \frac{0.9246}{\tau^{1/2}} \quad (18)$$

This result, plotted as a straight dotted line on figure 2, is correct to within 2 percent when  $\tau = 0.145$  and becomes increasingly accurate as  $\tau$  decreases. For large times, the heat transfer approaches the steady-state condition asymptotically. However, a practical measure of the duration of the transient can be obtained from the time required for  $q$  to approach to within 5 percent of its steady-state value. From figure 2, the 5-percent time is found to be

$$\tau_{5\%} = 1.7 \quad \text{or} \quad t_{5\%} = \frac{1.7}{u_1} \quad (19)$$

An informative rephrasing of equation (19) may be achieved by introducing  $U_\infty$  from equation (3b). Then,

$$t_{5\%} = 1.7 \frac{x}{U_\infty} \quad (19a)$$

Now,  $x/U_\infty$  is the time required for fluid moving at velocity  $U_\infty$  to traverse the distance  $x$ . Although, as expected,  $t_{5\%}$  is somewhat in excess of  $x/U_\infty$ , these times are still of the same order of magnitude. A similar result has also been found for thermal step functions in laminar and turbulent pipe flows (see refs. 3 and 4).

The step function is the most rapid type of wall temperature change, and it would not be expected that quasi-steady conditions would prevail throughout the transient period. However, as a matter of curiosity, the quasi-steady heat transfer will be computed. This computation is carried out by using the steady-state relation for the heat-transfer coefficient in conjunction with the instantaneous temperature difference. From its definition  $h_{ss} = q_{ss}/(T_w - T_\infty)$ , the steady-state heat-transfer coefficient may be obtained from equation (16) as

$$h_{ss} = 0.4799 \frac{k}{(v/u_1)^{1/2}} \quad (20)$$

With this, the quasi-steady heat transfer may be determined by evaluating the expression  $q_{qs} = h_{ss}(T_w - T_\infty)_{inst}$ , from which it follows that

$$q_{qs} = 0.4799 \frac{k}{(v/u_1)^{1/2}} (T_w - T_\infty)_{inst} \quad (21)$$

For the step change in wall temperature,  $T_w - T_\infty$  is constant throughout the transient period and, as a consequence, the quasi-steady heat transfer is identical to the steady-state heat transfer. Hence, for the step jump, the ordinate of figure 2 also represents the ratio  $q/q_{qs}$ . It is easily seen that quasi-steady conditions are not achieved during the transient period.

Before leaving the step-function case, it is of interest to inquire as to whether comparisons are possible between the present results and those of previous analyses. Inasmuch as there has been no prior study of the transient period, such comparisons are only possible at the limits of small and large times. For exceedingly small values of time, it has been already noted that heat conduction dominates over convection. For this condition, the heat-transfer prediction of the present analysis is given by equation (18), which may be rephrased as

$$\frac{q}{k(T_w - T_\infty)} = \frac{0.4437}{(tv)^{1/2}} \quad (22)$$

The exact solution (ref. 9) for the heat-transfer response of a solid body (no velocities) to a step change in surface temperature may be phrased in a form identical to equation (22), with the numerical constant taking a value of 0.472035 for  $Pr = 0.7$ . Thus, the present prediction lies 6 percent low for this early period. At the other extremity of time, for the steady-state condition, the heat-transfer result of reference 10 based on a numerical solution of the differential energy equation<sup>2</sup> takes the same form as equation (16) herein, except that the

<sup>2</sup>This is in contradistinction to the integrated energy equation used here.

constant is 0.4958. So, the current prediction is low by about 3 percent. This level of accuracy is sufficient for almost all applications.

#### GENERALIZATION TO ARBITRARY TIME-DEPENDENT WALL TEMPERATURE

The linearity of the energy equation permits the use of a superposition technique to generalize the step-function results. Consider a process in which there is no heat transfer (i.e.,  $T_w = T_\infty$ ) for  $t < t^*$ ; and then, at  $t = t^*$ , a step change in wall temperature  $dT_w$  is applied. The heat-transfer response to such a process may be computed from equation (15) to be

$$dq = \frac{3}{2} k dT_w \frac{1}{\Delta_{t-t^*}} \quad t \geq t^* \quad (23)$$

where the notation  $\Delta_{t-t^*}$  is used to indicate that  $\Delta$  is a function of  $t - t^*$  rather than of  $t$  (since the transient starts at  $t = t^*$  instead of  $t = 0$ ). But, as may be seen by referring to figure 3, this small step may be considered as an elementary part of an arbitrarily variable wall temperature. The heat-transfer response to such a time-dependent wall temperature is found by integrating equation (23), which gives

$$q = \frac{3}{2} k \int_0^t \frac{dT_w, t^*}{\Delta_{t-t^*}} \quad (24a)$$

or

$$q \frac{(v/u_1)^{1/2}}{k} = 0.5649 \int_0^\tau \frac{dT_w, \tau^*}{\phi_{\tau-\tau^*}} \quad (24b)$$

where dimensionless variables have been introduced in equation (24b). In these equations,  $T_w$  is taken as a function of the dummy integration variable  $\tau^*$  (or  $t^*$ ), and, hence, a second subscript is used. The notation  $\phi_{\tau-\tau^*}$  indicates that  $\phi$  is to be regarded as a function of  $\tau - \tau^*$ ; and, for the purposes of the integration, the abscissa variable of figure 1 ought to be replaced by  $\tau - \tau^*$ . As written, equations (24a) and (24b) apply to any temperature variation, including step changes.<sup>3</sup> However, if there are no finite step changes in  $T_w$ , equation (24b) becomes

<sup>3</sup>Mathematically speaking, the integrals are Stijes integrals.

$$\frac{q(v/u_1)^{1/2}}{k} = 0.5649 \int_0^\tau \left( \frac{dT_{w,\tau^*}/d\tau^*}{\phi_{\tau-\tau^*}} \right) d\tau^* \quad (24c)$$

To illustrate the use of these expressions, an example will be given in a later section.

Thus far, consideration has been given to transients which begin from a condition of no heat transfer (i.e.,  $T_w = T_\infty$ ). The situation where the transient begins from an already established steady-state heat transfer  $q_0$  is also handled by superposition. By incorporating  $q_0$  into equation (24a), it follows that

$$q = q_0 + \frac{3}{2} k \int_0^t \frac{dT_{w,t^*}}{\Delta_{t-t^*}} \quad (25a)$$

or

$$\frac{q(v/u_1)^{1/2}}{k} = 0.5649 \left[ \frac{(T_w - T_\infty)_0}{1.177} + \int_0^\tau \frac{dT_{w,\tau^*}}{\phi_{\tau-\tau^*}} \right] \quad (25b)$$

Equations (24) and (25) give, at any time  $\tau$ , the heat transfer associated with a temperature variation over the interval from  $\tau = 0$  to  $\tau = \tau$ . On the other hand, the quasi-steady heat-transfer prediction at time  $\tau$  is computed from equation (21) as

$$q_{qs} = 0.4799 \frac{k}{(v/u_1)^{1/2}} (T_w - T_\infty)_\tau \quad (26)$$

With this, and with equation (25b), the ratio of the instantaneous to the quasi-steady heat transfer takes the form

$$\frac{q}{q_{qs}} = \frac{(T_w - T_\infty)_0}{(T_w - T_\infty)_\tau} + \frac{1.177}{(T_w - T_\infty)_\tau} \int_0^\tau \frac{dT_{w,\tau^*}}{\phi_{\tau-\tau^*}} \quad (27)$$

This expression may be used to determine those times during the transient period at which the state is essentially quasi-steady (e.g., within 5 or 2 percent).

## HEAT-TRANSFER RESULTS FOR LINEARLY VARYING WALL TEMPERATURE

To illustrate the use of the generalized heat-transfer results, consideration is given to the case of a wall temperature which varies linearly with time, that is,

$$T_w - T_\infty = A^* \tau^* \quad (28)$$

Here  $\tau^*$  has been used as the running time variable in anticipation of application to equation (25b). Differentiation of equation (28) gives  $dT_{w,\tau^*} = A^* d\tau^*$ . With this and  $(T_w - T_\infty)_0 = 0$ , equation (25b) becomes

$$\frac{q(v/u_1)^{1/2}}{k} = 0.5649 A^* \int_0^\tau \frac{d\tau^*}{\phi_{\tau-\tau^*}} \quad (29)$$

For the integration, it is convenient to introduce a new dummy variable  $\xi$  by the definition:  $\xi = \tau - \tau^*$ , from which it follows that

$$\frac{q(v/u_1)^{1/2}}{k} = 0.5649 A^* \int_0^\tau \frac{d\xi}{\phi_\xi} \quad (29a)$$

The notation  $\phi_\xi$  means that as a consequence of the transformation,  $\phi$  is now a function of the single variable  $\xi$ . For the purposes of integration, both figure 2 and equation (11) directly give the required relation between  $\phi$  and the running integration variable  $\xi$ .

The variation of the heat transfer with time, obtained from integrating equation (29a), is presented as the solid curve in figure 4. At very early times, the heat transfer is small because the driving force  $T_w - T_\infty$  is also small. The heat transfer increases monotonically with time because of the sustained increase of  $T_w - T_\infty$ . A steady-state heat-transfer condition is never achieved.

It is possible to derive simple asymptotic expressions for the results of figure 4. At early times, the variation of  $\phi$  with  $\xi$  is given by equation (12). Utilizing this, it is found from equation (29a) that

$$\frac{q(v/u_1)^{1/2}}{kA^*} = 0.8874 \tau^{1/2} \quad (30)$$

A plot of this relation is given as the dotted straight line in figure 4. It is found that, for  $\tau = 0.392$ , the deviation between the dotted line and the solid curve is 2 percent; and, as time decreases, this deviation diminishes. At the other extreme, for large times,  $\varphi$  becomes essentially constant. Then, in equation (29a), the integral can be conveniently subdivided into two parts in the following way:

$$\int_0^{\tau} \frac{d\xi}{\varphi_{\xi}} = \int_0^{\tau_1} \frac{d\xi}{\varphi_{\xi}} + \frac{\tau - \tau_1}{1.177}$$

where  $\tau_1$ , the time at which  $\varphi = 1.177$ ,<sup>4</sup> is found from equation (11) to be 6.522. The integral from 0 to  $\tau_1$  has been carried out numerically and has the value 6.459. With this information, equation (29a) yields the heat-transfer asymptote for large times as follows:

$$\frac{q(v/u_1)^{1/2}}{kA^*} = 0.5187 + 0.4799 \tau \quad (31)$$

This equation is plotted as the dashed line in figure 4. At  $\tau = 1.45$ , equation (31) is good to within 2 percent; and, with increasing time, the deviation becomes progressively smaller. It is interesting to observe that the two asymptotic curves form a rather tight bracket around the solid curve which is the exact representation of equation (29a).

It is of interest to compare the quasi-steady heat-transfer predictions with those of figure 4. The general expression for  $q/q_{qs}$  as given by equation (27) may be specialized to the following form for the particular case now under consideration:

$$\frac{q}{q_{qs}} = \frac{1.177}{\tau} \int_0^{\tau} \frac{d\xi}{\varphi_{\xi}} \quad (32)$$

The results obtained by integrating equation (32) have been plotted in figure 5. As expected, the greatest deviations between the instantaneous  $q$  and the quasi-steady value occur at the beginning of the transient. As time proceeds, quasi-steady conditions are approached more and more closely. Also shown in figure 5 are asymptotic lines appropriate to small and large values of time. The equations of these lines are as follows:

Small times: 
$$\frac{q}{q_{qs}} = \frac{1.849}{\tau^{1/2}} \quad (33a)$$

<sup>4</sup>This is sufficiently close to the steady-state value of 1.17705 to provide four figure results for the heat transfer.

Large times: 
$$\frac{q}{q_{qs}} = 1 + \frac{1.081}{\tau} \quad (33b)$$

The range of applicability is the same as that for the asymptotic lines of figure 4. From equation (33b), the time can be determined when essentially quasi-steady conditions are achieved. For most practical purposes, the state can be considered quasi-steady when  $q/q_{qs} = 1.05$ . This condition is achieved when

$$\tau_{5\%} = 21.65 \quad \text{or} \quad t_{5\%} = 21.65 \frac{x}{U_{\infty}} \quad (34)$$

Comparison with equation (19) shows that a much longer time is required to achieve quasi-steady conditions during a linear temperature rise than after a step jump. This is not surprising since in the former case the wall boundary condition is continually changing with time and requires a continuous pursuit by the flow; while, in the latter case, the boundary condition changes abruptly and thereafter remains constant.

#### PROCEDURE FOR RAPID CALCULATIONS

By inspection of heat-transfer relations (24) to (27), it is seen that computations for the case of an arbitrary time-dependent wall temperature require the knowledge of the boundary-layer thickness  $\phi$  for the step-function case. It is clear that these heat-transfer computations could be shortened if a simple analytical expression for  $\phi$  were available. A representation which immediately suggests itself is the dotted and dashed envelope curve of figure 1, the mathematical description of which is

$$\left. \begin{aligned} \phi &= 1.27303 \tau^{1/2} & 0 \leq \tau \leq 0.855 \\ \phi &= 1.17705 & \tau \geq 0.855 \end{aligned} \right\} \quad (35a)$$

In using these expressions, it must be realized that some sacrifice in precision is being made in order to achieve computational ease and speed. In terms of  $\tau - \tau^*$  as an independent variable, equation (35a) becomes

$$\left. \begin{aligned} \phi_{\tau-\tau^*} &= 1.27303(\tau - \tau^*)^{1/2} & 0 \leq \tau - \tau^* \leq 0.855 \\ \phi_{\tau-\tau^*} &= 1.17705 & \tau - \tau^* \geq 0.855 \end{aligned} \right\} \quad (35b)$$

With this approximation for  $\phi$ , the heat-transfer equation (25b) corresponding to an arbitrarily prescribed variation in  $T_w$  becomes

$$q \frac{(v/u_1)^{1/2}}{k} = 0.4799(T_w - T_\infty)_0 + 0.4437 \int_0^\tau \frac{dT_{w,\tau^*}}{(\tau - \tau^*)^{1/2}} \left\{ \begin{array}{l} (36a) \\ 0 \leq \tau \leq 0.855 \end{array} \right.$$

$$q \frac{(v/u_1)^{1/2}}{k} = 0.4799(T_w - T_\infty)_{\tau-0.855} + 0.4437 \int_{\tau-0.855}^\tau \frac{dT_{w,\tau^*}}{(\tau - \tau^*)^{1/2}} \left\{ \begin{array}{l} (36b) \\ \tau \geq 0.855 \end{array} \right.$$

where the notation  $(T_w - T_\infty)_{\tau-0.855}$  indicates that  $T_w - T_\infty$  is evaluated at  $\tau^* = \tau - 0.855$ . In deriving equation (36b) from (25b), cognizance has been taken of the fact that  $\phi$  is constant as  $\tau^*$  ranges from 0 to  $\tau - 0.855$ . The ratio of the instantaneous to the quasi-steady heat transfer as given by equation (27) may also be modified by introducing the simplified expressions for  $\phi$ , which leads to the form

$$\frac{q}{q_{qs}} = \frac{(T_w - T_\infty)_0}{(T_w - T_\infty)_\tau} + \frac{0.9246}{(T_w - T_\infty)_\tau} \int_0^\tau \frac{dT_{w,\tau^*}}{(\tau - \tau^*)^{1/2}} \left\{ \begin{array}{l} (37a) \\ 0 \leq \tau \leq 0.855 \end{array} \right.$$

$$\frac{q}{q_{qs}} = \frac{(T_w - T_\infty)_{\tau-0.855}}{(T_w - T_\infty)_\tau} + \frac{0.9246}{(T_w - T_\infty)_\tau} \int_{\tau-0.855}^\tau \frac{dT_{w,\tau^*}}{(\tau - \tau^*)^{1/2}} \left\{ \begin{array}{l} (37b) \\ \tau \geq 0.855 \end{array} \right.$$

To illustrate the application of these simplified heat-transfer relationships, attention is directed to the case of a linearly varying wall temperature as described by equation (28). By noting that

$$\left. \begin{array}{l} dT_{w,\tau^*} = A^* d\tau^* \\ (T_w - T_\infty)_0 = 0 \quad (T_w - T_\infty)_{\tau-0.855} = A^*(\tau - 0.855) \end{array} \right\} \quad (38)$$

it is found by direct integration of equation (36) that

$$\frac{q(v/u_1)^{1/2}}{k} = 0.8874 \tau^{1/2} \quad 0 \leq \tau \leq 0.855 \quad (39a)$$

$$\frac{q(v/u_1)^{1/2}}{k} = 0.4799 \tau + 0.4103 \quad \tau \geq 0.855 \quad (39b)$$

These heat-transfer results are plotted in figure 6 as a dot-dashed curve. Also shown there for comparison purposes is a solid curve representing the results of the more complete calculation which has already been described in a previous section of the report and in figure 4. It is seen that the performance of the rapid computational procedure is quite satisfactory; the maximum deviation from the solid curve is only 7 percent, while the deviations over most of the range are much smaller. The relationship between the instantaneous and quasi-steady heat transfer for the linearly varying wall temperature may be computed from equation (37) after introducing (38). By direct integration there is obtained

$$\frac{q}{q_{qs}} = \frac{1.849}{\tau^{1/2}} \quad 0 \leq \tau \leq 0.855 \quad (40a)$$

$$\frac{q}{q_{qs}} = 1 + \frac{0.8549}{\tau} \quad \tau \geq 0.855 \quad (40b)$$

There is no need to plot these results since their percentage deviations from the results of the complete computation are identical to those already noted with relation to the heat-transfer results of figure 6. From equation (40b), the time at which  $q/q_{qs} = 1.05$  is found to be

$$\tau_{5\%} = 17.1 \quad \text{or} \quad t_{5\%} = 17.1 \frac{x}{U_\infty} \quad (41)$$

This is in reasonable agreement with the value of 21.65 previously obtained.

The satisfactory performance of the rapid computation procedure for the case of the linearly varying wall temperature helps to establish confidence in its accuracy. But, even with this, it is worthwhile to apply it with caution in situations where the time-dependence of  $T_w$  is very irregular.

## APPLICATION OF RESULTS

## Application to Unsteady Heating or Cooling of a Wall

Now, attention is turned to demonstrating the facility with which the results may be applied. In many situations, the time-variation of the wall temperature might not be prescribed. Rather, it would be determined by the balance between the rate of convective heat transfer to the wall and the capacity of the wall to absorb this energy. For the simplest situation of high-conductivity wall material and no heat losses, the temperature rise of the wall is given by

$$\frac{dT_w}{dt} = bq \quad (42)$$

where  $b$  includes the wall specific heat, density, and thickness. Changing notation from  $t$  to  $\tau^*$  and inserting in equation (25b) yield an integral equation for  $q$ , solution of which would provide the variation of the heat-transfer rate with time. Then, the variation of the wall temperature could be found from equation (42).

Further complications could be introduced to account for wall conduction, heat losses, and so forth. However, these details need not be considered here.

## Application to Variable Properties and Viscous Dissipation

The analysis given in this report has been built on the supposition of an incompressible, constant-property, nondissipative flow. It is desirable to have some way of determining the heat transfer when compressibility, et cetera are present. To make a complete analysis of this problem would be a very formidable task. However, on the basis of what has been done here, it is possible to offer a simple approach for obtaining a first estimate of the heat transfer to these more complicated flows. The procedure is as follows: By using the given wall temperature information, the ratio  $q/q_{qs}$  is computed from equation (27) as a function of time. Then, the quasi-steady heat transfer would be evaluated not from equation (26), but rather by applying the steady-state heat-transfer coefficients appropriate to the compressible, variable-property, dissipative problem. Then, with this value of  $q_{qs}$  in conjunction with the result of equation (27), it is expected that a fair estimate of  $q$  could be obtained.

Lewis Research Center

National Aeronautics and Space Administration  
Cleveland, Ohio, July 22, 1959

## APPENDIX - SOLUTION OF EQUATION (8) BY PARTIAL FRACTIONS

The first step in attacking equation (8) is to determine the six factors of the polynomial which appears in the denominator, the roots of which are

$$\left. \begin{aligned} \varphi_1 &= 9.44638 & \varphi_2 &= 1.17705 \\ \varphi_{3,4} &= 2.68534 \pm 2.82741 i \\ \varphi_{5,6} &= -0.536598 \pm 0.729478 i \end{aligned} \right\} \quad (A1)$$

With these, the left-hand side of the equation can be written as

$$\frac{\varphi d\varphi}{(\varphi - \varphi_1)(\varphi_1 - \varphi_2) \dots (\varphi - \varphi_6)} = \left[ \frac{A}{(\varphi - \varphi_1)} + \frac{B}{(\varphi - \varphi_2)} + \dots + \frac{F}{(\varphi - \varphi_6)} \right] d\varphi \quad (A2)$$

where a partial fraction expansion has been used to obtain the last form. The constants A, B, . . . , F are found by the usual methods. To illustrate, attention is focused on finding A. First, equation (A2) is multiplied through by  $\varphi - \varphi_1$ , which gives

$$\frac{\varphi}{(\varphi - \varphi_2)(\varphi - \varphi_3) \dots (\varphi - \varphi_6)} = A + B \frac{\varphi - \varphi_1}{\varphi - \varphi_2} + \dots + F \frac{\varphi - \varphi_1}{\varphi - \varphi_6} \quad (A3)$$

Now, equations (A2) and (A3) must apply for all values of  $\varphi$  and, in particular, for  $\varphi = \varphi_1$ . For this choice, equation (A3) becomes

$$A = \frac{\varphi_1}{(\varphi_1 - \varphi_2)(\varphi_1 - \varphi_3) \dots (\varphi_1 - \varphi_6)} \quad (A4)$$

which is an arithmetic expression for A. The constants B, C, . . . , F are determined in an analogous fashion.

The end result of these computations is exhibited in equation (10). Since the complex roots are conjugates, the partial fraction expansion may be simplified by combining terms. In this way, the  $\varphi - \varphi_3$  and  $\varphi - \varphi_4$  terms of the expansion combine to form the third term of equation (10); while  $\varphi - \varphi_5$  and  $\varphi - \varphi_6$  give rise to the fourth term of that equation.

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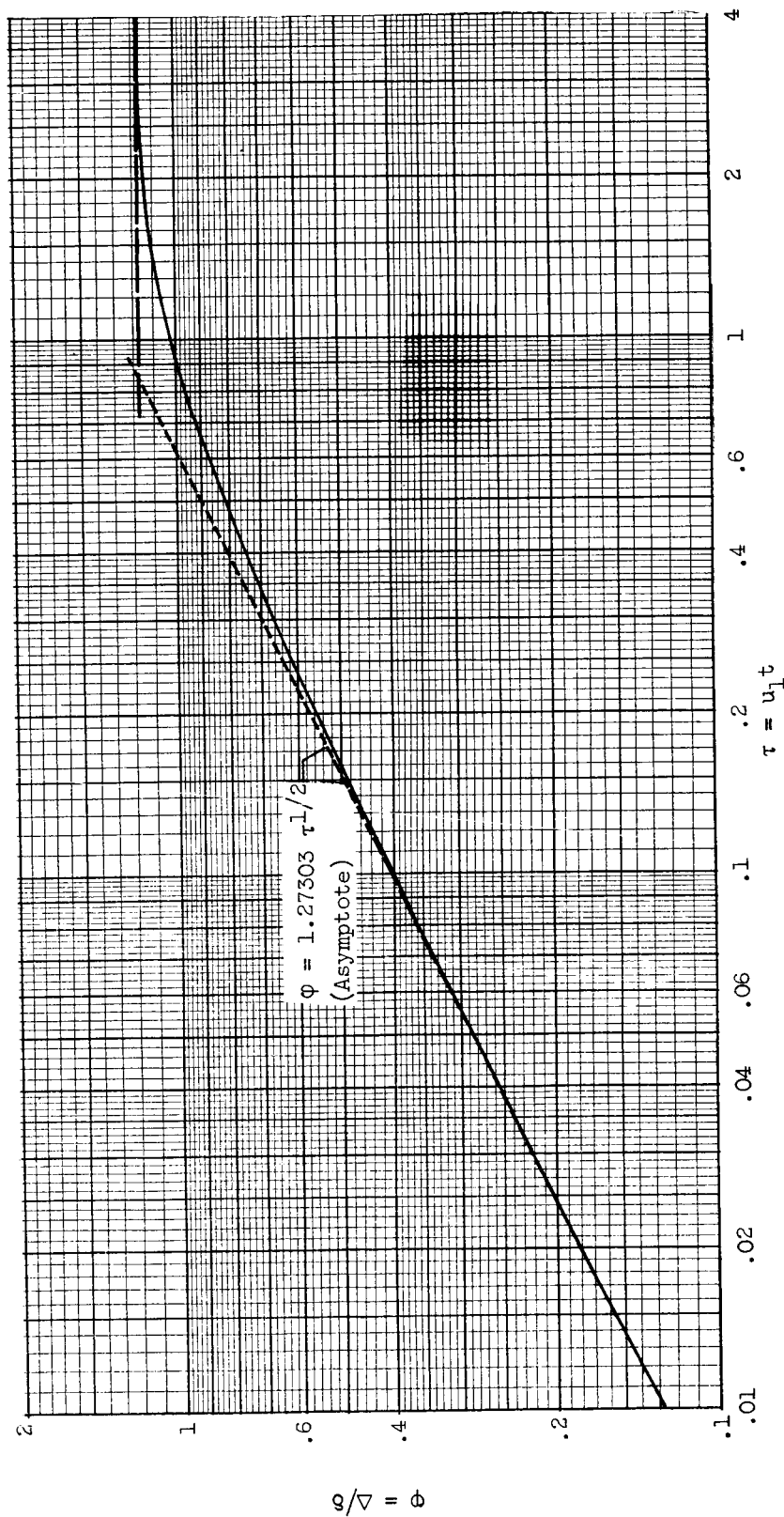


Figure 1. - Growth of thermal boundary layer after a step jump in wall temperature.  
 $\delta = \text{const.} = (7.052 \nu / u_1)^{1/2}$ .

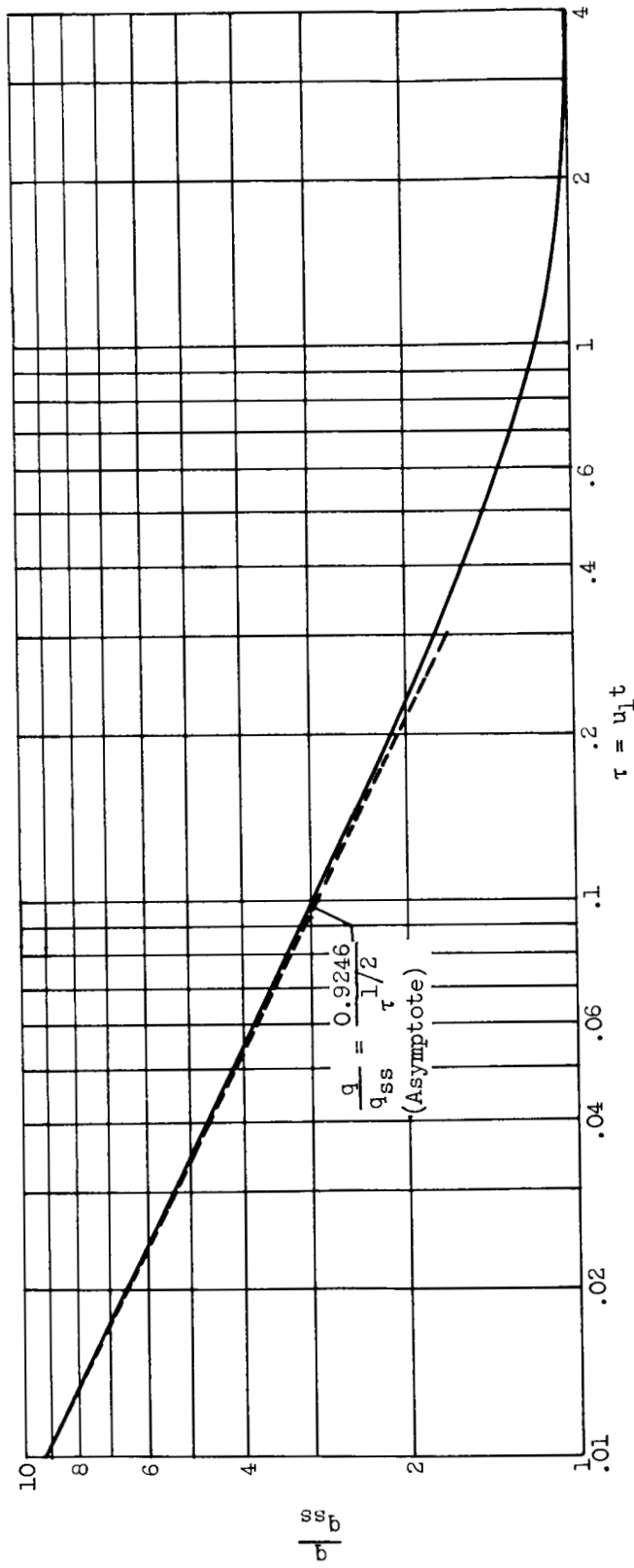


Figure 2. - Heat-transfer response to a step jump in wall temperature.

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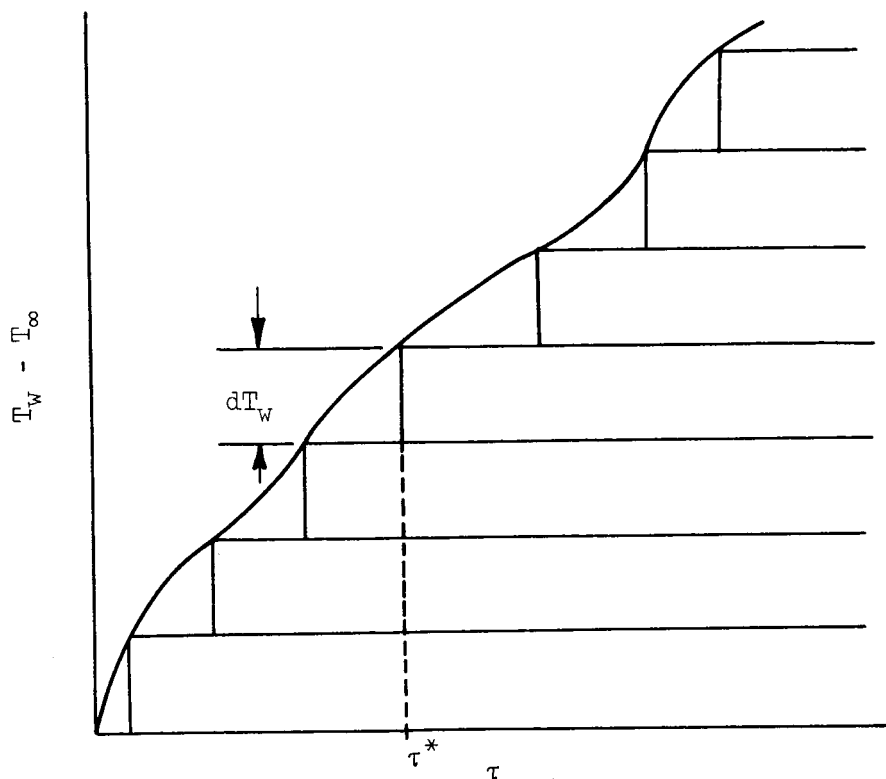


Figure 3. - Representation of arbitrary time variation in wall temperature.

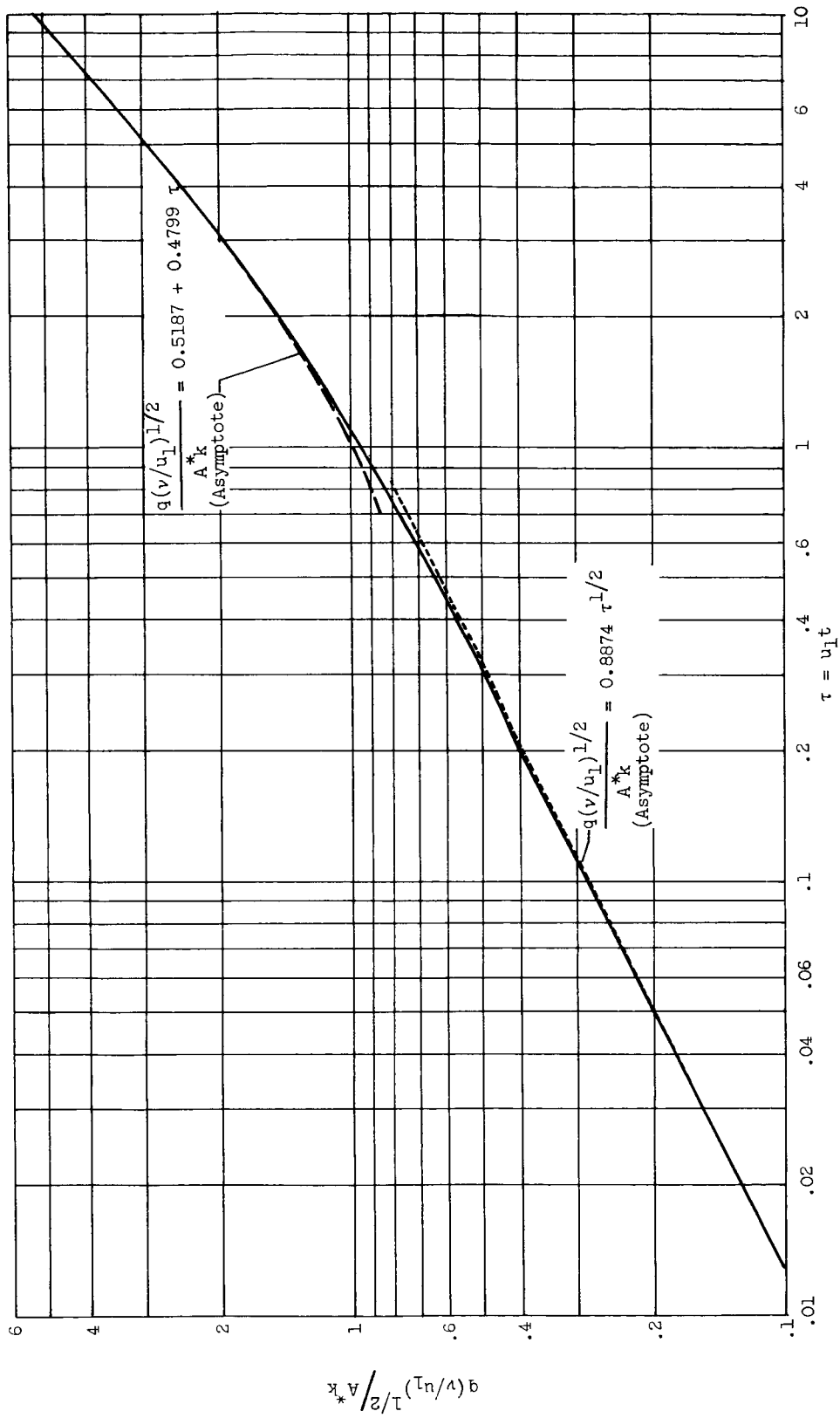


Figure 4. - Heat-transfer response to a linearly varying wall temperature  $T_w - T_\infty = A^* \tau$ .

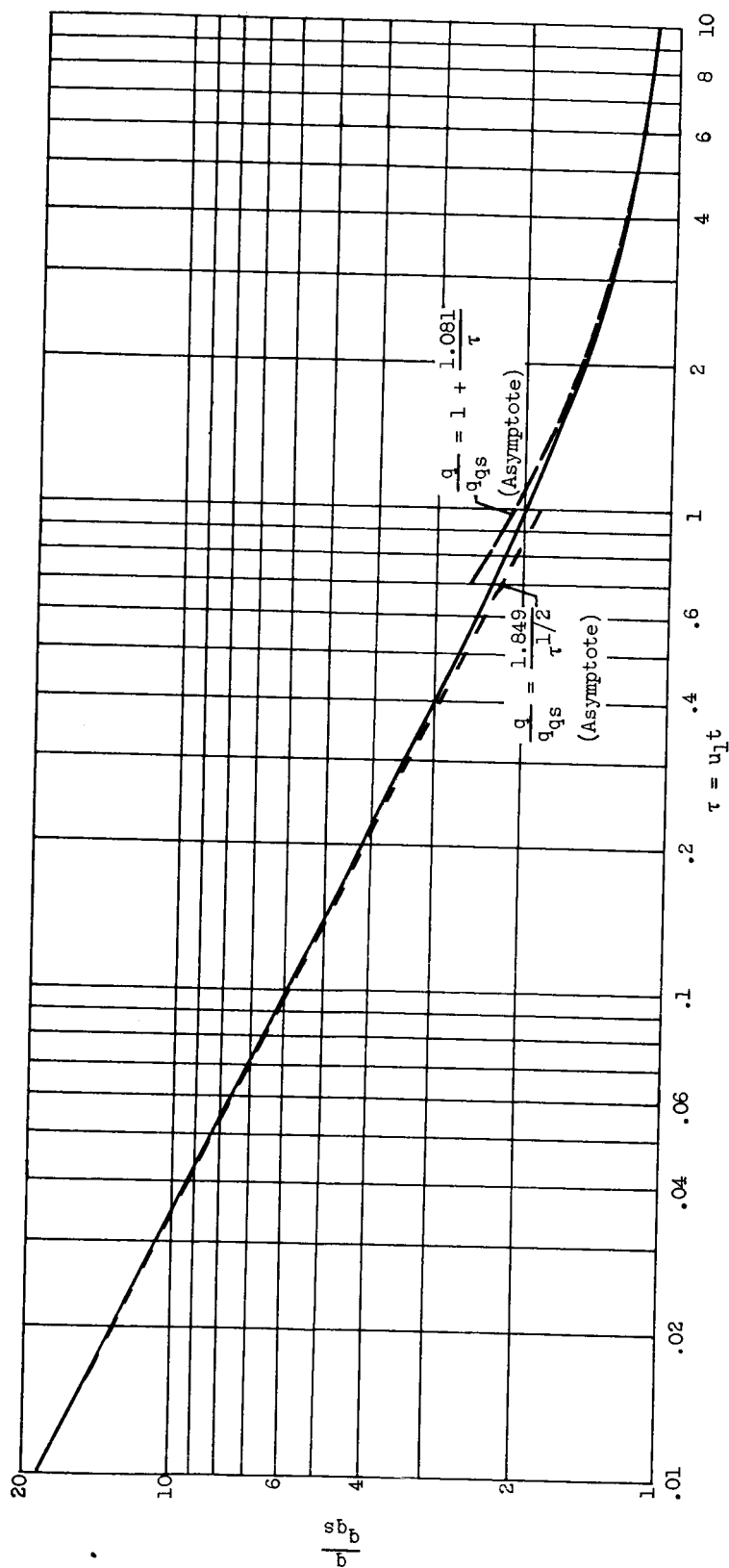


Figure 5. - Comparison of instantaneous and quasi-steady heat transfer for a linearly varying wall temperature.

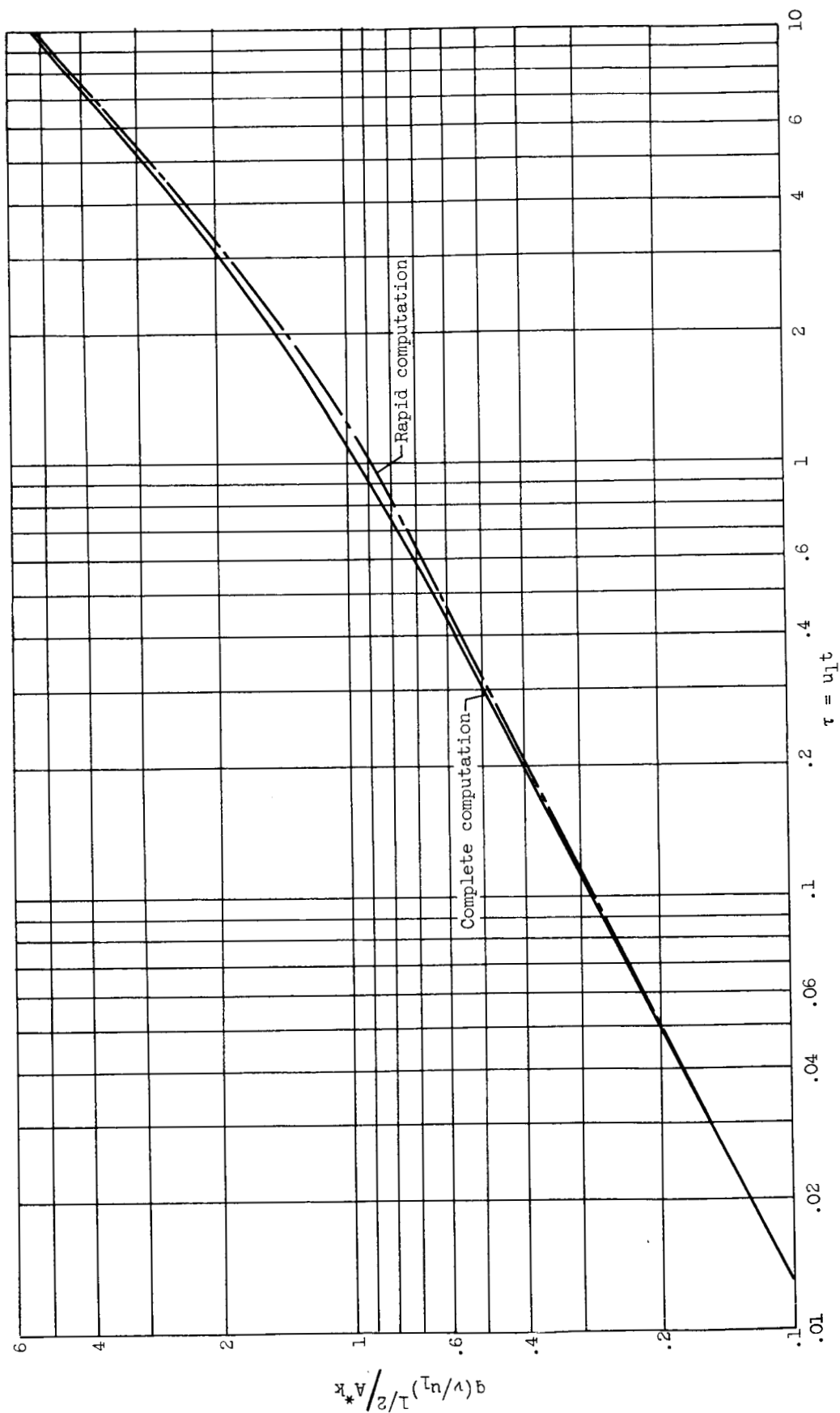


Figure 6. - Comparison of heat-transfer results for linearly varying wall temperature.